AN ASYMPTOTIC METHOD FOR ANALYSIS OF FINITE AMPLITUDE OSCILLATIONS IN PARALLEL FLOWS

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A Thesis Submitted
In Partial Fulfilment of the Requirements
for the Degree of
MASTER OF TECHNOLOGY

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By

S. NATARAJAN

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CERTIFICATE

Certified that this work has been carried out under my supervision and that this has not been submitted elsewhere for a degree.

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S. Natarajan

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SYNOPSIS

This thesis deals with the effect of finite amplitude oscillations on the stability of parallel flows at large Reynolds numbers. It is shown that matched asymptotic expansions can be fruitfully used in analysing these effects. Two limit processes are required to describe the behaviour of the disturbance. One limit describes it away from the critical layer where the phase velocity of the disturbance is equal to the undisturbed velocity. The effects of viscosity and nonlinearity are of a higher order in this region. In the critical layer, viscosity and nonlinearity are significant to the lowest order. Since the resulting governing equations are slightly modified versions of Raleigh's equation and Tollmien's equation, the results of the linear theory can be readily used.

The results of preliminary calculations of neutral stability curve for the lowest order effects and the eigenfunction compare favourably with the results calculated by other methods.

The advantages of the present method are (a) simplification of the equations and hence reduction of computation time (b) a clear indication that the nonlinear effects are confined to the critical layer to a first approximation, (c) ready application of the results of the linear theory to the higher order theory.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

The classical hydrodynamic analysis deals with conditions under which small disturbances can grow and the manner in which they lead to more complex laminar or turbulent flows. The linear analysis valid for small disturbances describes the behaviour in the early phase of its growth. Nonlinear analysis describing the later stages have been employed by Stuart (1960), Watson (1960), Eckhaus (1965) and Reynolds and Potter (1967).

The interest in nonlinear stability stems from various considerations. One objective is to obtain insight in the transition from laminar to turbulent flows. A second objective is to develop the analysis so that it can be applied with modification to a turbulent flow. This objective arises from the success of earlier uses by Phillips (1967), and Landahl (1967) in describing certain properties of turbulent flows, by adopting the analysis of classical stability theory.

The mathematical problem of hydrodynamic stability is formulated by taking the given steady flow and superimposing a disturbance of suitable kind. This results in a set of nonlinear disturbance equations which govern the behaviour of the disturbance. If the solution of the equation shows that any disturbance ultimately decays to zero the flow is said to be stable. If the disturbance can be permanently different from zero, it is unstable. Instability sometimes leads to turbulent flow and sometimes to another possibly more complex laminar flow.

The analysis of the nonlinear stability theory leads to the equation for the amplitude of the velocity disturbance in the form

$$dA/dt = f\ (A) = a_0 A + a_2 A^3 + \ldots\ldots. \qquad (1.1)$$

The linear stability theory yields the constant a_0 as the eigenvalue of the Orr-Sommerfeld problem. The aim of the nonlinear theory is to determine the remaining a_n, particularly a_2.

If $a_0 < 0$, the flow is stable to infinitesimally small disturbances. But the question remains as to whether a disturbance of sufficient amplitude produces instability. Such subcritical instability arises if the higher order terms outweigh the a_0 term.

If $a_0 > 0$, the flow is unstable to infinitesimally small disturbances but it is possible that the higher order terms may balance the leading term a_0. Then a supercritical equilibrium flow may be obtained.

In case $a_0 = 0$ which corresponds to the neutral stability curve of the linear theory the sign of a_2 determines whether the disturbance actually grows or decays.

In order to determine a_0, a_2 etc, two types of analyses have been formulated one by Stuart (1960) and Watson (1960) and the second by Eckhaus (1965).

In the work of Stuart (1960), a two dimensional disturbance of the travelling wave form with its higher harmonics is assumed which, when substituted in the flow equation yields a system of equations for the harmonics. Approximations based on the order

of magnitude considerations yield a finite system of equations. Watson's (1960) analysis is so constructed that Stuart's analysis appears as the lowest order approximation.

Eckhaus' (1965) method which deals with a class of nonlinear stability problems, consists in expanding the disturbance function in terms of the eigenfunctions of the linearized disturbance equation. Qualitatively this method, when applied to plane poiseuille flow leads to the results of Stuart.

The main objective of the thesis is to develop a method of matched asymptotic expansions for the nonlinear problem to bring out the close resemblance of the higher order problem to the lowest order problem. The analysis of Stuart and Watson has been followed in the thesis. The specific problem of two dimensional disturbances of a plane poiseuille flow is considered. The method with minor modifications can presumably applied to three dimensional disturbances such as those considered by Reynolds and Potter (1967) and to other parallel flows.

Application to boundary layer type of flows is more involved and presents additional problems of dealing with normal velocity which is of no great consequence in linear theory. The analysis is also restricted to the case where $a_o = 0$. This restriction was made in earlier analyses and is also present in Reynolds and Potter (1967) computations with which the numerical results obtained here are compared.

The second chapter describes the analysis of Stuart, Watson

and Reynolds and Potter, and also the asymptotic analysis of the first order and the higher order problems. The third chapter contains the results of numerical computation and discussion of the results.

CHAPTER 2

ANALYSIS OF FINITE AMPLITUDE DISTURBANCE IN PARALLEL FLOWS

2.1 PRELIMINARIES

The Stuart-Watson (1960) analysis of finite amplitude disturbances dealt with two-dimensional disturbances. In a subsequent paper Stuart (1961) considered the interaction of a two-dimensional disturbance with a particular three-dimensional wave. Reynolds (1967) extended the method of Stuart and Watson to include a class of three-dimensional disturbances.

In a two-dimensional linear analysis, a key assumption is that the perturbation stream function ψ representing the departure of the flow field from the basic steady laminar flow can be represented by harmonic components of the form $\psi = \phi(y)\exp[i\alpha(x-ct)]$ In a nonlinear analysis, the stream function is expanded in terms of the basic Orr-Sommerfeld wave, keeping in view, the three important effects of nonlinearity on the motion.

They are (1) the interaction of the basic wave with itself to produce a mean Reynold's stress which distorts the mean velocity field, (2) the possibility that there may be subcritical instability or supercritical equilibrium flows, and (3) possible modification of the wave velocity of the disturbance. Any expansion or perturbation scheme should allow for all these three effects. This has been achieved in the Stuart-Watson-Reynolds formalism.

Starting with the governing vorticity transport equation

$$\zeta_{,t} + \psi_{,y}\zeta_{,x} - \psi_{,x}\zeta_{,y} = R^{-1}\nabla^2\zeta \qquad (2.1.1)$$

where $\zeta = \nabla^2\psi$, a comma followed by a suffix indicates partial differentiation. Here all the lengths are nondimensionalised with respect to the half channel width h, time with respect to h/U_0 where U_0 is the centre line velocity and the stream function with respect to (hU_0).

In the undisturbed laminar flow, the motion is parallel to the planes and is given by

$$U = 1 - y^2 \qquad (2.1.2)$$

In a linear analysis of disturbances, the perturbation would be assumed to be of the form $\psi = \phi(y) \exp[i(\alpha x + \omega t)] \exp(at)$ and $(\omega - ia)$ would emerge as the complex eigenvalue. The term $\exp(at)$ would describe the behaviour of disturbance amplitude with time and the stability would be determined by the sign of a. In a nonlinear analysis, a solution in terms of this basic wave and its harmonics is sought which would reflect the effects of nonlinearity mentioned earlier. Therefore the frequency of the basic wave ω is taken to be dependent on the amplitude A (t) and the stream function is described in terms of intermediate variables, amplitude A and phase θ.

Let $\omega = \omega(A)$, and $\theta = \alpha x + \omega t$ (2.1.3)

Then (2.1.4)

$$dA/dt\,\zeta_{,A} + [\omega + \omega_{,A}(t\,dA/dt) + \psi_{,y}]\zeta_{,\theta} - \psi_{,\theta}\zeta_{,y} - R^{-1}(\zeta_{,yy} + \alpha^2\zeta_{,\theta\theta}) = 0$$

The stream function is then expanded in terms of harmonics of the basic wave as

$$\psi(A,y,\theta) = \psi^{(k)}(A,y)e^{ik\theta} + \bar{\psi}^{(k)}(A,y)e^{-ik\theta} \qquad (2.1.5)$$

Here the summation convention given in appendix has been used.

If the amplitude of the disturbance is small the harmonics $\psi^{(k)}$ may be expanded in a power series of A, such that 0(A) expansion corresponds to the linear analysis. That is

$$\psi^{(k)} = A^{n} \phi^{(k;n)} \qquad (2.1.6)$$

In addition the amplitude growth rate and the frequency dependence on amplitude is assumed to be

$$dA/dt = a_{n} A^{n} \qquad (2.1.7)$$

and $\omega + t\, d\omega/dt = b_{n} A^{n}$ (2.1.8)

with these substitutions, the differential equation governing the behaviour of $\phi^{(k;n)}$, which represents that portion of the k-th harmonic $\psi^{(k)}$ of the stream function which has an amplitude of $0(A^{n})$, is (Reynolds and Potter 1967)

$$L_{kn} \phi^{k;n} = i\alpha c^{[n-1]} G \delta_{k1} + N_{kn} \qquad (2.1.9)$$

where

$$L_{kn} = ik\alpha\left[(U-c)(D^{2}-k^{2}\alpha^{2}) - D^{2}U + 1/ik\alpha R\,(D^{2}-k^{2}\alpha^{2})^{2}\right]$$

$$i\alpha c^{(n)} = -(a^{(n)} + ib^{(n)})$$

$$G = (D^{2}-\alpha^{2})\phi^{(1;1)}$$

$$N_{kn} = -(ma^{[n-m]} + ikb^{[n-m]})(D^{2}-k^{2}\alpha^{2})\phi^{\{k;m\}} + F_{kn}/(1+\delta_{k0})$$

$$\begin{aligned} F_{kn} = -i\Big[& jD\phi^{[k-j;n-m]}(D^{2}-j^{2}\alpha^{2})\phi^{[j;m]} + \\ & (k+j)D\bar{\phi}^{[j,n-m]}(D^{2}-(k+j)^{2}\alpha^{2})\phi^{[k+j;m]} \\ & -jD\phi^{[k+j;n-m]}(D^{2}-j^{2}\alpha^{2})\bar{\phi}^{[j;m]} \\ & -(k-j)\phi^{[k-j;n-m]}D(D^{2}-j^{2}\alpha^{2})\phi^{[j;m]} \\ & +j\bar{\phi}^{[j;n-m]}D(D^{2}-(k+j)^{2}\alpha^{2})\phi^{[k+j;m]} \\ & -(k+j)\phi^{[k+j;n-m]}D(D^{2}-j^{2}\alpha^{2})\bar{\phi}^{[j;m]}\Big] \end{aligned}$$

The forcing function on the right side in (2.1.9) is the sum of a large number of terms and is written in a compact form using a summation convention explained in the appendix. For odd values of (k+m), $\phi^{(k;m)}$ is zero (Reynolds and Potter 1967).

This equation describes how the various harmonics interact among themselves in contributing a term of $O(A^m)$. The nonlinear influence of one harmonic on another is described by the forcing function in (2.1.9). For k = 1, m = 1 equation (2.1.9) reduces to the Orr-Sommerfeld equation. $\phi^{(0;2m)}$ represents the distortion of the mean flow velocity due to nonlinear effects.

This formalism enables one to solve equation (2.1.9) successively. The calculation of the eigenvalues $c^{(n)}$ can be found in Reynolds and Potter (1967). Earlier computations were by direct numerical methods. The presence of small parameter in the operator L_{km} creates difficulties in numerical work which are wellknown from the experience with Orr-Sommerfeld equation.

An alternate approach of expansion of the solution in terms of eigenfunctions of Orr-Sommerfeld equations was used by Pekeris and Shkoller (1969) but it does not seem to reveal the essential nature of the disturbance directly as a large number of terms are required and 8th to 10th terms start becoming numerically more important than 2nd, 3rd or 4th. It is not easy to interpret such behaviour. Examination of the results, say the distortion of average profile due to nonlinear interactions indicates that the basic structure of higher order problem is similar to lowest order problem and it can be possibly revealed by using asymptotic methods.

2.2 FIRST ORDER ANALYSIS

The first order analysis is taken up now as it provides a simple indication about the nature of higher order analysis. The method of matched asymptotic expansions used here lead to the wellknown classical results (Lin 1955). The method does not require any specialized schemes as the classical analytical continuation method does. Furthermore the method can be employed even when the initial velocity profile is not analytic.

The first order disturbance equation is the wellknown Orr-Sommerfeld equation;

$$(U-c)(D^2-\alpha^2)\phi^{(1;1)} - D^2U\,\phi^{(1;1)} + i/\alpha R\,(D^2-\alpha^2)^2\phi^{(1;1)} = 0 \qquad (2.2.1)$$

where $D \equiv d/dy$.

This equation of transport of disturbance vorticity is obtained by putting $k = n = 1$ in equation (2.1.9).

The boundary conditions are

$$y = \pm 1, \quad \phi^{(1;1)} = D\phi^{(1;1)} = 0 \qquad (2.2.1a)$$

Since U is symmetrical about y-axis any solution of Orr-Sommerfeld equation can be split into even and odd parts. It is known that even solution becomes unstable first. Therefore it is convenient to consider even solution over the range (0,−1) and replace the boundary condition at $y = +1$ by the following condition of symmetry.

$$y = 0, \quad D\phi^{(1;1)} = D^3\phi^{(1;1)} = 0 \qquad (2.2.1b)$$

All higher derivatives of the form $D^{(2n+1)}\phi^{(1;1)}$ vanish as a consequence of equation (2.2.1).

Two limits of the governing equations are considered. In the outer limit, $R \to \infty$ for a fixed y. In the inner limit, $R \to \infty$ for fixed $(y-y_c)/(\alpha U_c' R)^{-1/3}$, y_c and α. The subscript c indicates the value of a quantity where $U = C$. The former limit describes the behaviour outside a thin layer (critical layer) located at y_c, and the latter describes the behaviour within the critical layer.

The outer and inner expansions are taken as

$$\phi^{(1;1)} = \phi_0^{(1;1)} + R^{-1}\phi_1^{(1;1)} + o(R^{-1})$$

where $\phi_n^{(1;1)}$ is a function of y, α and C and

$$\phi^{(1;1)} = \psi_0^{(1;1)} + \delta \ln \delta\, \psi_1^{(1;1)} + \delta \psi_2^{(1;1)} + o(\delta)$$

In the classical analysis (Stuart 1963), the gage function of $O(\delta \ln \delta)$ was omitted. But it is necessary for matching of the inner and outer expansions.

Here $\psi_\eta^{(1;1)}$ is a function of η, α, and C, $\eta = (y-y_c)/\delta$, $\delta = (\alpha U_c' R)^{-1/3}$.

The outer limit gives rise to the Raleigh's equation for the lowest order term.

$$\left[(U-C)(D^2-\alpha^2) - D^2 U\right] \phi_0^{(1;1)} = 0 \qquad (2.2.2)$$

The above equation states that, as the effects of viscosity are of a higher order, vorticity of the disturbed flow is conserved. Its solution can be written as

$$\phi_0^{(1;1)} = E\psi_1 + F\psi_2 \qquad (2.2.3)$$

where $\psi_j = (U-C)\sum_{r=0} (k\alpha)^{2r} I_r(f_j)$

And $f_1 = 1$, $f_2 = \int_b^y (U-C)^{-2}\,dy$, $I_0(f_j) = f_j$

$I_n(f_j) = \int_{y_c}^{y} (U-C)^{-2}\,dy \int_{y_c}^{y} (U-C)^2 I_{n-1}(f_j)\,dy$, $n>0$, $j=1,2$

Here h is a suitable point in the interval $(y_c, 0)$.

The inner limit gives rise to the following equations, which we will refer to as Tollmien's equations.

$$L\,\psi_i^{(1;1)} = 0, \quad i = 0,1 \qquad (2.2.4)$$

$$L\,\psi_2^{(1;1)} = i a_c \left(\tfrac{1}{2}\eta^2 D^2 \psi_0^{(1;1)} - \psi_0^{(1;1)}\right) \qquad (2.2.5)$$

where a_c is u_c''/u_c'. D and L are operators defined by

$$D \equiv d/d\eta \quad \text{and} \quad L \equiv d^4/d\eta^4 - i\eta\, d^2/d\eta^2$$

The above equations state that even in the lowest order approximation, the viscous diffusion of vorticity has to be considered in the critical layer.

The solution of these equations is

$$\psi_i^{(1;1)} = c_{i1} + c_{i2}\eta + c_{i3} h_1(\eta) + c_{i4} h_2(\eta) + \delta_{i2} F(\eta) \qquad (2.2.6)$$

$i = 0,1,2$ and δ_{i2} is the Kronecker's delta, $F(\eta)$ is the particular integral of (2.2.5), and

$$h_i(\eta) = \int_{a_i}^{\eta}\int_{a_i}^{\eta'} \eta^{1/2} H^{(i)}_{1/3}\left[2/3\,(i\eta)^{3/2}\right] d\eta\, d\eta', \quad i = 1,2$$

$$a_1 = +\infty, \quad a_2 = -\infty$$

The outer solution is subjected to the following conditions arising from (2.2.1b).

$$y = 0, \quad D\phi_0^{(1;1)} = D^3\phi_0^{(1;1)} = 0 \qquad (2.2.7)$$

If the outer solution satisfies either of the above, it follows from (2.2.2) that the other is also satisfied. Thus symmetry imposes only one condition on the outer solution.

The inner solution is subjected to the following condition at the lower wall

$$\eta = \eta_w = -(1+y_c)/\delta, \; \phi^{(1;1)} = d\phi^{(1;1)}/d\eta = 0 \qquad (2.2.8)$$

Additional conditions required to determine the constants of integration arise from matching.

MATCHING

The main idea in the method of matched asymptotic expansions is that, although the inner and outer limits provide good approximations in different regions, there is an overlapping region, where both are good approximations. So we consider the behaviour of the disturbance as described by the outer limit and examine it near the critical layer for small $\bar{y} = (y-y_c)$.

$$(U-c) = u_c'\bar{y}\,(1+a_c\bar{y}/2 + o\,(\bar{y})),\quad a_c = u_c''/u_c'$$

$$I_o\,(f_1) = 1,\quad I_n\,(f_1) = o\,(\bar{y})\ \text{for}\ n \geq 1$$

$$I_o\,(f_2) = u_c'^{-2}\left[-1/\bar{y} - a_c\ln\bar{y} + J\,(\bar{y})\right]$$

$$I_1\,(f_2) = O\,(\bar{y}),\quad I_n\,(f_2) = o\,(\bar{y})\ \text{for}\ n > 1$$

$$J\,(\bar{y}) = u_c'^{2}\int_{\bar{b}}^{\bar{y}}\left[(U-c)^{-2} - u_c'^{-2}\,(1/\bar{y}^2 - a_c/\bar{y})\right]\,d\bar{y} + 1/\bar{b} + a_c\ln\bar{b}$$

$$= J_c + O\,(\bar{y})$$

hence

$$\phi_c^{(1;1)} = -F/u_c' - (Fa_c/u_c')\,\bar{y}\ln\bar{y} + \bar{y}(Bu_c' - Fa_c/2u_c' + FJ_c/u_c') + o(\bar{y})$$

The above form indicates why the outer solution is not a good approximation within the critical layer. As $\bar{y} \to 0$, $\phi_c^{(1;1)}$ has a plausible behaviour but $d\,\phi_c^{(1;1)}/d\bar{y} \sim \ln\bar{y}$. This means that tangential velocity would become infinite if the outer solution were valid within the critical layer. The outer solution can also be written in the inner variable η as

$$\phi_e^{(1;1)} = -F/u_e' - \delta \ln \delta \, (Fa_e/u_e')\eta + \delta \left[\eta \, (Ba_e - Fa_e/Ba_e' + BJ_e/u_e') - (Fa_e/u_e') \, \eta \ln \eta \right] + o(\delta\eta) \qquad (2.2.9)$$

Matching with the inner solution (2.2.6) to the order $\delta \ln \delta$, we get

$c_{01} = -F/u_e'$, $c_{02} = 0$, $c_{04} = 0$, $c_{11} = 0$, $c_{12} = -a_e F/u_e'$, $c_{14} = 0$

since h_2 is an exponentially increasing function for large .

As h_1 is an exponentially decreasing function no restriction is placed on c_{03} and c_{13}.

Matching to the order δ requires the behaviour of the particular integral $P(\eta)$ associated with the forcing function in equation (2.2.6) which is $-ia_e c_{01} + ia_e c_{03} \, (\frac{1}{2}\eta^2 h_1'' - h_1)$. If $N(\eta)$ and $r(\eta)$ are the particular integrals arising out of the forcing functions $-i$ and $a_e i \, (\frac{1}{2}\eta^2 h_1'' - h_1)$, it can be shown that $N(\eta) \sim \eta \ln \eta$ and $r(\eta)$ goes to zero exponentially as $\eta \to \infty$. Hence $P(\eta) \sim a_e c_{01} \, \eta \ln \eta$ as $\eta \to \infty$.

Matching to the order δ gives

$c_{21} = 0$, $c_{22} = Ba_e' - Fa_e/Ba_e' + BJ_e/u_e', -a_e c_{01} - Fa_e/u_e'$, $c_{24} = 0$. There is no restriction on c_{23} since h_1 decays exponentially. Note that the third condition is automatically satisfied in view of the earlier condition on c_{01}. The inner solution can therefore be written as

$$\psi_0^{(1;1)} + \delta \ln \delta \, \psi_1^{(1;1)} + \delta \psi_2^{(1;1)} = -F/u_e' - \delta \ln \delta \, (Fa_e/u_e')\eta + \delta \left[\eta \, (Ba_e - Fa_e/Ba_e' + BJ_e/u_e') - (Fa_e/u_e') \, N(\eta) \right] + (c_{03} + \delta \ln \delta \, c_{13} + \delta \, c_{23}) \, h_1(\eta) + \delta \, c_{03} r(\eta) \qquad (2.2.10)$$

EIGENVALUE PROBLEM

In general, equation (2.2.1) with its boundary conditions, gives rise to a characteristic value problem resulting in a condition of the type

$F(\alpha, R, C) = 0$ where F is a complex valued function.

For each pair of real values of α and R there is a characteristic value C. For real values of C, which means that the disturbances are neutrally stable, this relation leads to a curve in the (α, R) plane. This curve is called the neutral stability curve.

Equations (2.2.8) and (2.2.10) may be written, when the boundary conditions (2.2.7) and (2.2.8) are applied, as follows

$$Ef_1'(0) + Ff_2'(0) = 0 \qquad (2.2.11)$$

$$Eg_1(\eta_w) + Fg_2(\eta_w) + Gh_1(\eta_w) + \delta e_{02} r(\eta_w) = 0 \qquad (2.2.12)$$

$$Eg_1'(\eta_w) + Fg_2'(\eta_w) + Gh_1'(\eta_w) + \delta e_{02} r'(\eta_w) = 0 \qquad (2.2.13)$$

f_1, f_2, g_1, g_2 are written for the terms occurring in equations (2.2.8) and (2.2.10), $D = e_{02} + \delta \ln \delta \, e_{12} + \delta e_{22}$, and η_w is the value of η at the lower wall.

The above conditions imply that the critical layer extends to the wall. Earlier work based on this assumption has received experimental support.

Notice that D and e_{02} are of the same order. Since the function $r(\eta)$ arises as a particular integral of (2.2.8), when the forcing function is due to h_1, its behaviour for large negative η is related

to the behaviour of h_1. For a given η_w, r may be assumed to be comparable to h_1, hence the term $\delta\, e_{w3} r(\eta_w)$ would be much smaller than $Bh_1(\eta_w)$. This approximation is implicit in earlier work. With this approximation (2.2.11), (2.2.12) and (2.2.13) become homogeneous equations in E, F and D. Nontrivial solution requires that,

$$\begin{vmatrix} f_1'(0) & f_2'(0) & 0 \\ g_1(\eta_w) & g_2(\eta_w) & h_1(\eta_w) \\ g_1'(\eta_w) & g_2'(\eta_w) & h_1'(\eta_w) \end{vmatrix} = 0 \qquad (2.2.14)$$

This can be reduced to an equation of the form

$$F(\eta_w) = E(\alpha, c) \qquad (2.2.15)$$

where $F(\eta_w) = h_1(\eta_w)/\eta_w h_1'(\eta_w)$

and $$E(\alpha, c) = 1+\{u_c''-u_c'/(1+y_c)\}/(J_c u_c'-u_c''(3/2+\ln(1+y_c)-i\pi) - f_2'(0)/f_1'(0)u_c'^2)$$

This equation when solved yields the values of η_w, α, c from which the Reynolds number R can also be calculated.

EIGENFUNCTION

Knowing the eigenvalue c for a given α and R, the eigenfunction $\phi^{(1;1)}$ can be calculated using the outer expansion from y = 0 upto a point very near the critical point and then using the inner expansion. The function is normalised in such a way that $\phi^{(1;1)} = 1$ at the axis. The calculated eigenfunction is shown in fig.2.

2.2 HIGHER ORDER ANALYSIS

The higher order analysis follows closely the first order analysis. That is, the effect of viscosity and nonlinearity are of a higher order in the region outside the critical layer and in the critical layer they are significant to the lowest order. The forcing functions in equation (2.1.9), therefore, are expected to be significant in the critical layer. Let $\phi^{(k;m)} \sim R^{-a_m}$. The following relation for a_m can be found so that the above hypothesis is satisfied. Which is,

$a_m = a_{m-n} + a_n - 2/3$ for all $n = 1, 2 \ldots (m-1)$

This reduces to $a_m = m(r-2/3) + 2/3$ where $r = a_1$.

Thus $\phi^{(k;m)} = R^{-[m(r-2/3)+2/3]}\psi^{(k;m)} + \ldots$

Further to make the terms involving $a^{(m)}$, $b^{(m)}$ and $c^{(m)}$ of the same order as the other terms on the right hand side of equation (2.1.9), an assumption regarding eigenvalues of higher order functions $\phi^{(1;m)}$ is made.

i.e. $\{a^{(m)}; b^{(m)}; c^{(m)}\} = R^{-[m(r-2/3)+1/3]}\{A^{(m)}; B^{(m)}; C^{(m)}\}$

Substitution in the equation (2.1.9) gives,

$$L_{km}\psi^{[k;m]} = i\alpha C\delta_{k1}^{[m-1]}(D^2-\alpha^2)\psi^{(1;1)}R^{-1/3}R^{-2/3}(mA^{m-n} + iRB^{[m-n]})x$$
$$(D^2-k^2\alpha^2)\psi^{\{k;m\}} + R - 2/3\, f_{km}/(1+\delta_{k0}) \qquad (2.2.1)$$

As before, the outer expansion is assumed to be

$\psi^{(k;m)} = \phi_0^{(k;m)} + R^{-1}\phi_1^{(k;m)} + o(R^{-1})$

and the inner expansion is taken to be

$\psi^{(k;m)} = \Psi_0^{(k;m)}(\eta) + \delta\ln\delta\,\Psi_1^{(k;m)}(\eta) + \delta\Psi_2^{(k;m)}(\eta)$

where $\delta = (\alpha u_c' R)^{-1/3}$, $\eta = (y-y_c)/\delta$.

The outer limit $R \to \infty$ for fixed y gives

$$(U-c)(D^2 - k^2\alpha^2)\psi_0^{(k;n)} - D^2U\,\psi_0^{(k;n)} = 0, \quad k \neq 0 \qquad (2.3.2)$$

The above equation shows that higher order effects and higher harmonics are governed by a modified Rayleigh's equation outside the critical layer. Thus the nonlinear forcing function and viscous terms do not enter in the lowest order approximation.

The inner limit $R \to \infty$ for fixed η gives

$$\begin{aligned}
L_k\psi_0^{(k;n)} = {} & iku_c'\left(\tfrac{1}{2}\eta^2 D^2\psi_0^{(k;n)} - \psi_0^{(k;n)}\right)\delta_{n2} \\
& -\beta^2 i\alpha c^{[n-1]} D^2\psi_0^{(1;1)}\delta_{k1} \\
& +\beta^2\left(mA^{[n-m]} + ikB^{[n-m]}\right)D^2\psi_0^{(k;m)} \\
& - i\beta/(1+\delta_{m0})(1+\delta_{k0})\left[j\left(D\psi_0^{[k-j;n-m]}D^2\psi_0^{[j;m]}\right.\right. \\
& \left.+ D\psi_0^{[k-j;n-m]}D^2\psi_0^{[j;m]}\right) - j\left(D\psi_0^{[k+j;n-m]}D^2\bar{\psi}_0^{[j;m]}\right. \\
& \left.+ D\psi_0^{[k+j;n-m]}D^2\bar{\psi}_0^{[j;m]}\right) + (k+j)\left(D\bar{\psi}_0^{[j;m]}D^2\psi_0^{[k+j;n-m]}\right. \\
& \left.+ D\bar{\psi}_0^{[j;m]}D^2\psi_0^{[k+j;n-m]}\right) - j\left(\psi_0^{[j;m]}D^3\psi_0^{[k-j;n-m]}\right. \\
& \left.+ \psi_0^{[j;m]}D^3\psi_0^{[k-j;n-m]}\right) - (k+j)\left(\psi_0^{[k+j;n-m]}D^3\bar{\psi}_0^{[j;m]}\right. \\
& \left.+ \psi_0^{(k+j;n-m)}D^3\bar{\psi}_0^{[j;m]}\right) + j\left(\bar{\psi}_0^{[j;m]}D^3\psi_0^{[k+j;n-m]}\right. \\
& \left.\left.+ \bar{\psi}_0^{[j;m]}D^3\psi_0^{[k+j;n-m]}\right)\right]
\end{aligned} \qquad (2.3.3)$$

where $L_k = d^4/d\eta^4 - ik\eta\, d^2/d\eta^2$, $D = d/d\eta$, $\beta = (\alpha u_c')^{-1/3}$

$\psi^{(k;n)}$ is odd for even n and even for odd n.

The condition at the axis is therefore

$$y = 0 \quad D\psi^{(k;n)} = 0 \quad \text{for } n \text{ even}$$
$$\psi^{(k;n)} = 0 \quad \text{for } n \text{ odd.}$$

OUTER SOLUTION

The solution of equation (2.3.2) can be obtained as

$$\phi_o^{(k;n)} = X^{(k;n)}\Psi_1 + Y^{(k;n)}\Psi_2 \qquad (2.3.4)$$

where $\Psi_j = (U-c)\sum_{\nu=0} (k\alpha)^{2\nu} I_\nu(f_j)$

where $I_\nu(f_j)$ is the same as in the previous section. Applying the symmetry condition we get

$$X^{(k;n)}\Psi_1(0) + Y^{(k;n)}\Psi_2(0) = 0 \quad \text{for } n \text{ odd}$$
$$X^{(k;n)}\Psi_1'(0) + Y^{(k;n)}\Psi_2'(0) = 0 \quad \text{for } n \text{ even} \qquad (2.3.5)$$

Since the behaviour of $I_\nu(f_j)$ is known in the limit $y \to y_c$ (or $\bar{y} \to 0$), we get $\Psi_o^{(k;n)} = X^{(k;n)} u_c'\bar{y} + Y^{(k;n)}/u_c'(J_c\bar{y} - 1 - a_c\bar{y}\ln\bar{y} - a_c\bar{y}/2) + o(\bar{y})$

This can be written in the inner variable as

$$\Psi_o^{(k;n)} = -Y^{(k;n)}/u_c' - \delta\ln\delta\,(Y^{(k;n)}a_c/u_c')\eta$$
$$+ \delta[\eta(X^{(k;n)}u_c' - Y^{(k;n)}a_c/2u_c' + Y^{(k;n)}J_c/u_c')$$
$$- (Y^{(k;n)}a_c/u_c')\eta\ln\eta] + o(\delta\eta) \qquad (2.3.6)$$

It is seen that the behaviour of $\Psi_o^{(k;n)}$ as $\bar{y} \to 0$ is similar to $\phi_o^{(1;1)}$ and does not involve k.

INNER SOLUTION

Since $\psi_o^{(k;n)}$ satisfies the equation (2.3.3) which is similar to Tollmien's equation, its solution is seen to be

$$\psi_s^{(k;n)} = c_{s1}^{(k;n)} + c_{s2}^{(k;n)}\eta + c_{s3}^{(k;n)} h_1(\eta;k) + c_{s4}^{(k;n)} h_2(\eta;k)$$
$$+ p_s^{(k;n)} + \delta_{s2}\, a_e c_{s1}^{(k;n)} H(\eta;k) \qquad (2.3.7)$$

where $h_j(\eta;k) = \int_{a_j}^{\eta}\int_{a_j}^{\eta'} \sqrt{\eta''}\, H^{(j)}_{1/3}\left[2/3\,(ik^{1/3}\eta'')^{3/2}\right] d\eta''\, d\eta'$

$j = 1, 2 \quad a_1 = \infty,\ a_2 = -\infty$

$H(\eta;k)$ is the particular solution of $L_k H(\eta;k) = -i$ and

$p_s^{(k;n)}$ is a particular integral given by

$$d^2 p_s^{(k;n)}/d\eta^2 = \pi/3k\left[h_2(\eta;k)\int_{\infty}^{\eta} h_1(\eta';k)\, r_s^{(k;n)}\, d\eta'\right.$$
$$\left. - h_1(\eta;k)\int_{-\infty}^{\eta} h_2(\eta';k)\, r_s^{(k;n)}\, d\eta'\right]$$

where $r_s^{(k;n)}$ is the forcing term in equation (2.3.3).

The boundary condition at the lower wall is

$$\eta = \eta_w = -(1+y_e)/\delta,\quad \psi^{(k;n)} = d\psi^{(k;n)}/d\eta = 0 \qquad (2.3.8)$$

MATCHING

The particular integrals $p_s^{(k;n)}$ go to zero in the limit $\eta \to \infty$ and therefore do not contribute to matching. The matching of the other terms gives

$c_{s1}^{(k;n)} = -f^{(k;n)}/u_e',\ c_{s3}^{(k;n)} = 0,\ c_{s4}^{(k;n)} = 0$

$c_{11}^{(k;n)} = 0,\ c_{12}^{(k;n)} = -f^{(k;n)} a_e/u_e',\ c_{14}^{(k;n)} = 0$

$c_{21}^{(k;n)} = 0,\ c_{22}^{(k;n)} = g^{(k;n)} u_e - f^{(k;n)} a_e/2u_e' + f^{(k;n)} f_e/u_e'$

$c_{24}^{(k;n)} = 0.$

The inner solution can therefore be written as

$$\psi^{(k;n)} = E^{(k;n)} u_e' \eta^{\delta} - F^{(k;n)}/u_e' \left[1 + a_e \eta^{\delta}(\ln\delta + \tfrac{1}{2}) + a_e \delta\, H(\eta;k) - J_e \eta^{\delta}\right] + D^{(k;n)} h_1(\eta;k) \qquad (2.3.9)$$

where $D^{(k;n)} = e_{e3}^{(k;n)} + \delta \ln\delta \; e_{13}^{(k;n)} + \delta e_{23}^{(k;n)}$

When the boundary conditions (2.3.8) are applied, equation (2.3.9) may be written in the form

$$E^{(k;n)} g_1(\eta_w) + F^{(k;n)} g_2(\eta_w) + D^{(k;n)} h_1(\eta_w;k) = P^{(k;n)}(\eta_w) \qquad (2.3.10)$$

$$E^{(k;n)} g_1'(\eta_w) + F^{(k;n)} g_2'(\eta_w) + D^{(k;n)} h_1'(\eta_w;k) = P'^{(k;n)}(\eta_w) \qquad (2.3.11)$$

where $P^{(k;n)} = P_e^{(k;n)} + \delta \ln\delta \; P_1^{(k;n)} + \delta P_2^{(k;n)}$

Equations (2.3.8), (2.3.10), (2.3.11) are sufficient to determine the constants $E^{(k;n)}$, $F^{(k;n)}$, $D^{(k;n)}$.

Eigenvalues of $\psi^{(1;n)}$

When k = 1, equations (2.3.8), (2.3.10) and (2.3.11) are of the form

$$E^{(1;n)} f_1'(0) + F^{(1;n)} f_2'(0) = 0 \qquad (2.3.12)$$

$$E^{(1;n)} g_1(\eta_w) + F^{(1;n)} g_2(\eta_w) + D^{(1;n)} h_1(\eta_w) = Q_1^{(1;n)} c^{[n-1]} + Q_2^{(1;n)} \qquad (2.3.13)$$

$$E^{(1;n)} g_1'(\eta_w) + F^{(1;n)} g_2'(\eta_w) + D^{(1;n)} h_1'(\eta_w) = Q_1'^{(1;n)} c^{[n-1]} + Q_2'^{(1;n)} \qquad (2.3.14)$$

where $c^{[n-1]}$ is the eigenvalue and $P^{(1;n)} = -(Q_1^{(1;n)} c^{[n-1]} + Q_2^{(1;n)})$

The coefficient matrix for this case is singular (equation 2.2.14). The condition for the solution of (2.3.12), (2.3.13), (2.3.14) to exist gives

$$c^{[n-1]} = \left(h_1(\eta_w)\, Q_2'^{(1;n)} - h_1'(\eta_w)\, Q_2^{(1;n)}\right) \Big/ \left(h_1'(\eta_w)\, Q_1^{(1;n)} - h_1(\eta_w)\, Q_1'^{(1;n)}\right) \qquad (2.3.15)$$

Thus the eigenvalues of higher order functions can be determined.

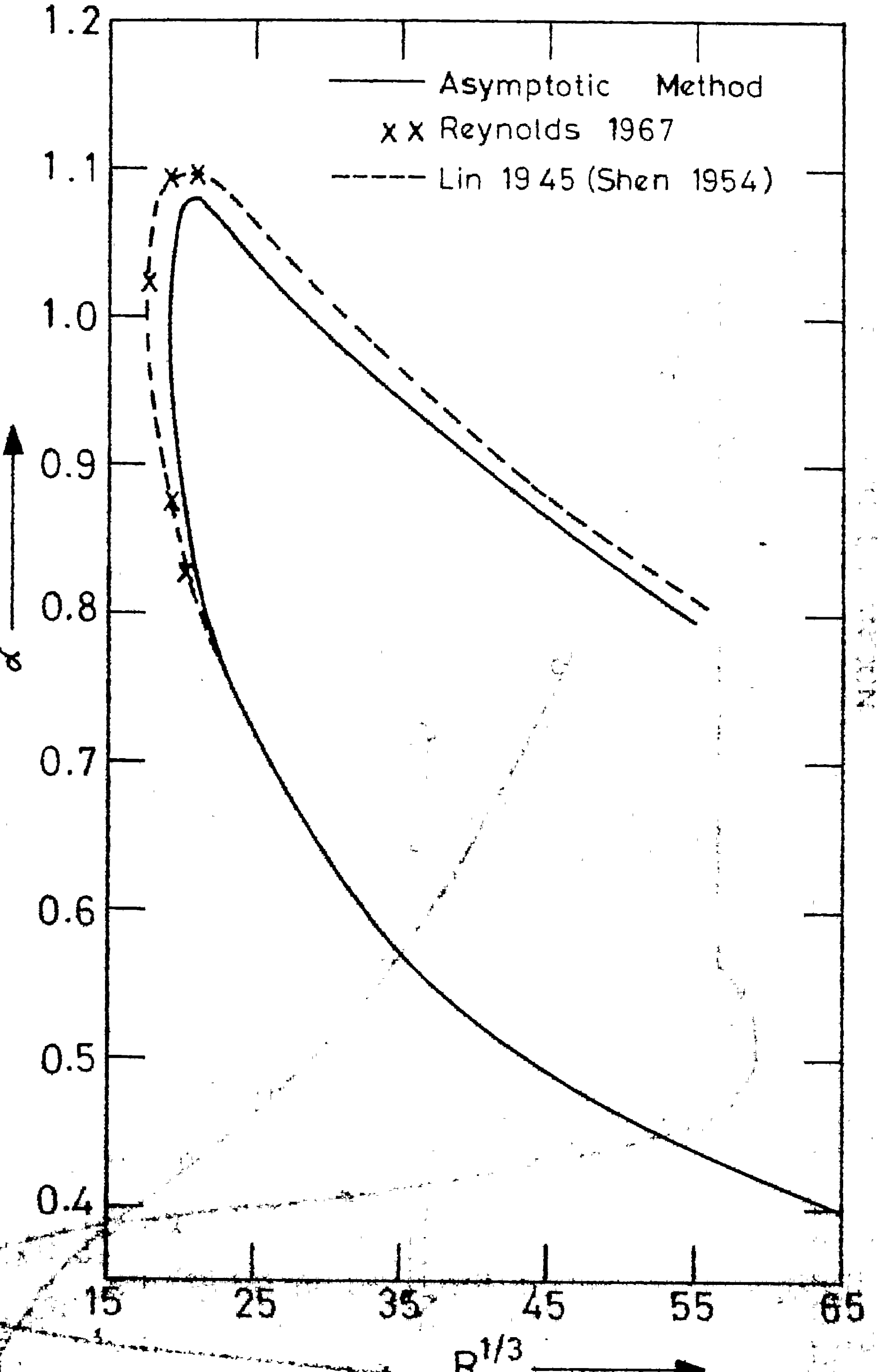
CHAPTER 3

RESULTS AND DISCUSSIONS

The asymptotic analysis presented in chapter 2 gives a procedure for evaluating the higher order functions and eigenvalues characterizing the effect of the nonlinear interaction of the disturbance. The method shows how the analysis of the first order disturbance equations can be extended to solve for the higher order functions.

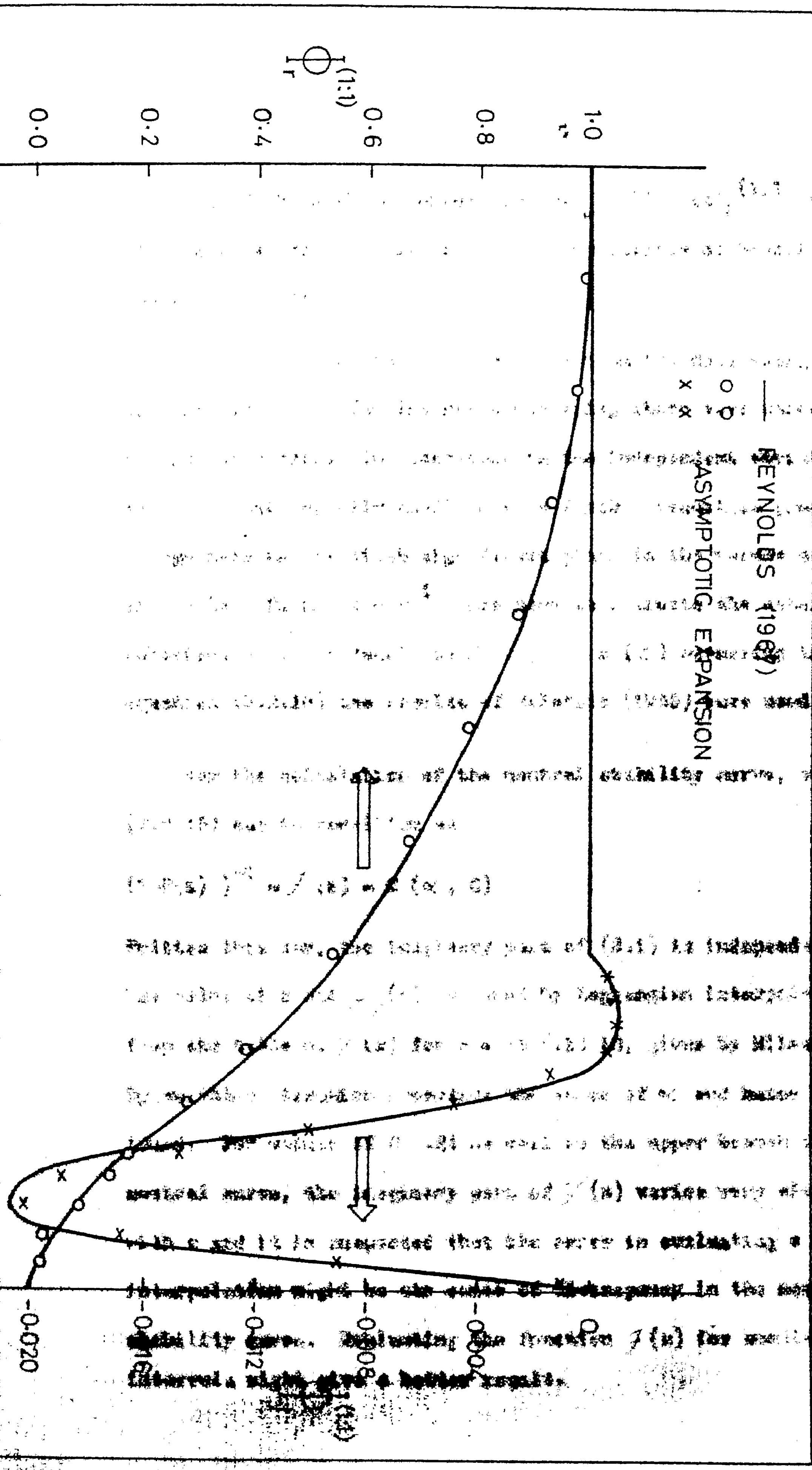
Calculations were carried out for the eigenvalue C of neutral disturbance over a wide range of disturbance wave numbers and Reynolds numbers. The eigenfunction for the critical value $R_c = 5772$, $C = .2640$ and $= 1.02$ was also calculated. The results are compared with those of Reynold's and Potter (1967) and Pekeris and Shkoller (1967), who have given the results of Lin (1946) and Shen (1954).

Fig. 1 shows the neutral stability curve computed by the method of chapter 2 and compares it with the earlier results. On the lower branch of the neutral stability curve, the asymptotic method gives the same results as other methods, but there is a small systematic disagreement on the upper branch and near the critical point.



Neutral stability curve

Fig.1



REYNOLDS (1967)
ASYMPTOTIC EXPANSION
1·0
0·8
0·6
0·4
0·2
0·0
$\phi_r^{(1:1)}$
0
-0·004
-0·008
-0·012
-0·016
-0·020
$\phi_i^{(1:1)}$

Fig. 3 shows the eigenfunction $\phi_r^{(1;1)} + i\phi_i^{(1;1)}$ and it is seen to compare well with the computations of Reynolds and Potter (1967).

The outer solution was calculated from the Heisenberg solution (2.2.8). The integrals occurring there were calculated by Simpson's rule. The increment in the independent variable was taken sufficiently small so that further reduction produced change only in the sixth significant place in the values of the integrals. Terms upto α^4 were used to evaluate the outer solution. For the functions $H(\eta)$ and $h(\eta)$ occurring in equation (2.2.10) the results of Holstein (1950) were used.

For the calculation of the neutral stability curve, equation (2.2.12) can be rewritten as

$$(1-F(z))^{-1} = \mathcal{F}(z) = E(\alpha, c) \tag{3.1}$$

Written this way, the imaginary part of (3.1) is independent of . The value of z and $\mathcal{F}_r(z)$ is found by Lagrangian interpolation from the table of $\mathcal{F}(z)$ for z = -6 (.1) 10, given by Miles (1960). By suitable iteration procedure the value of α and hence R may be found. For values of $c > .24$ as well as the upper branch of the neutral curve, the imaginary part of $\mathcal{F}(z)$ varies very slowly with z and it is suspected that the error in evaluating z by interpolation might be the cause of discrepancy in the neutral stability curve. Evaluating the function $\mathcal{F}(z)$ for smaller intervals might give a better result.

CHAPTER 4

CONCLUSIONS

1. The asymptotic method developed in chapter 2 brings out the similarity of the higher order problem to the lowest order problem, particularly the fact that the effect of viscosity and nonlinearity are confined to the critical layer.

2. The method can be extended to include the problem of three dimensional disturbances.

3. The outer and inner expansions are modified forms of Raleigh's and Tollmien's equation respectively. Thus the results of the linear theory can be readily used.

4. The numerical computation of the eigenfunction compares well with the results of Reynolds and Potter (1967), while that of the neutral stability curve shows a small systematic difference. A more refined numerical scheme may be required.

5. The numerical computation of the higher order functions, when completed, is expected to demonstrate the usefulness of the asymptotic method.

APPENDIX

SUMMATION CONVENTION

The summation convention used in chapter 2 is that of Reynolds and Potter (1967) and is given here.

The convention is that the terms are summed over all possible (integer) values of the repeated superscripts. Limits are placed on the values which the index can assume by punctuation as indicated below.

(n)	$n \geq 0,$
$[n]$	$n \geq 1$
$\{n\}$	$n \geq 2$
$m;n$	$n \leq m$

Negative indices are not permitted and hence zero is the lower bound. The delimiters ( ), [ ] and { } applied to a multiple superscript act only on the indices to the right of the semicolon:

$$[m;n] \quad \text{means} \quad 0 \leq n \leq m \geq 1$$

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